Black hole entropy without brick walls

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Abstract

The properties of the thermal radiation are discussed by using the new equation of state density motivated by the generalized uncertainty relation in the quantum gravity. There is no burst at the last stage of the emission of a Schwarzshild black hole. When the new equation of state density is utilized to investigate the entropy of a scalar field outside the horizon of a static black hole, the divergence appearing in the brick wall model is removed, without any cutoff. The entropy proportional to the horizon area is derived from the contribution of the vicinity of the horizon.

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The title is the same as Ref.[1] where Demers et al show that the divergence appearing in the brick wall model[2] can be absorbed into the renormalized Newton's constant. By using the WKB approximation, 't Hooft investigates the statistical properties of a scalar field outside the horizon of a Schwarzschild black hole. The entropy proportional to the horizon area is obtained, but with a cutoff utilized to remove the divergence of the density of states. The cutoff is introduced by hand and looks unnatural. Susskind and Uglum suggest that the explosive free energy and entropy in the model of 't Hooft are related to the divergence of the one-loop effective action of the quantum field theory in curved space[3]. Their conjecture is confirmed by [1]. The authors of [1] remove the cutoff and regularize the divergent free energy and entropy by introducing some regulators. These fictitious fields are especially designated in the number, statistics and masses. To my surprise, the entropy expressed by the masses of the regulators can be precisely renormalized to the Bekenstein-Hawking formula, $S = A/(4G_R)$, G_R is the renormalized Newton's constant. However, it is hard to understand the introduction of the " bare entropy" in Ref.[1]. The " bare entropy" seems to be negative and its meaning is unclear. Is there a better method can remove the divergence appearing in the brick wall model?

Recently, many efforts have been devoted to the generalized uncertainty relation

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2, \tag{1}$$

and its consequences[4]–[10], especially the effect on the density of states[9][10]. Here $\hbar$ is the Planck constant, λ is of order of Planck length. Eq. (1) means that there is a minimal length, $2\sqrt{\lambda}$. As well known, the number of quantum states in the integrals $d^3\mathbf{x}d^3\mathbf{p}$ is given by

$$\frac{d^3\mathbf{x}d^3\mathbf{p}}{(2\pi\hbar)^3}, \tag{2}$$

which can be understood as follows: since the uncertainty relation $\Delta x \Delta p \sim 2\pi\hbar$, one quantum state corresponds to a "cell" of volume $(2\pi\hbar)^3$ in the phase space. Based on the Liuville theorem, the authors of Ref.[10] argue that the number of quantum states should be changed to the following

$$\frac{d^3\mathbf{x}d^3\mathbf{p}}{(2\pi\hbar)^3(1+\lambda p^2)^3}, \tag{3}$$

where $p^2 = p_i p^i, i = 1, 2, 3$. Eq. (3) seriously deforms the Planckian spectrum of the black body radiation at the Planck temperature, $T_\lambda = \sqrt{1/\lambda}$ (see Ref.[10], fig.2).

let us discuss the more details than Ref.[10]. This will benefit the following investigation of the black hole entropy. From Eq.(3), we directly write down the density of internal energy of the thermal radiation

$$\begin{aligned} u &= \int_0^\infty \frac{\omega^3 d\omega}{(e^{\beta\omega}-1)(1+\lambda\omega^2)^3} \\ &= \beta^{-4}\int_0^\infty \frac{x^3 dx}{(e^x-1)(1+ax^2)^3} \\ &= \beta^{-4}G(a), \end{aligned} \tag{4}$$

where $a=\lambda/\beta^2, x=\beta\omega$. We take the units $G=c=\hbar=k_B=1$. The above integral can not be expressed as a simple formula, but we can investigate its asymptotic behavior in the two different conditions. We first consider the case $a \ll 1$. This means that the temperature is much less than the Planck temperature. We have

$$\begin{aligned} G(0) &= \int_0^\infty \frac{x^3 dx}{e^x-1} = \frac{\pi^4}{15}, \\ G'(0) &= -3\int_0^\infty \frac{x^5 dx}{e^x-1} = -\frac{24\pi^6}{63}, \end{aligned} \tag{5}$$

then

$$\begin{aligned} u &= \beta^{-4}[G(0)+G'(0)a] \\ &= \frac{\pi^4}{15}\beta^{-4}\left(1-\frac{40\pi^2\lambda}{7\beta^2}\right). \end{aligned} \tag{6}$$

In the usual case, above equation does not essentially change the well known conclusion for the black body radiation because the correction is very slight. For example, the temperature of the center of the neutron star is 10^9K, but the Planck temperature is $10^{32}K, \lambda/\beta^2 \sim 10^{-46}$. However, Eq. (6) is no longer valid for the case $\lambda/\beta^2 \gg 1$, that is higher than the Planck temperature. We calculate the upper bound of energy density, that is

$$\begin{aligned} u &< \beta^{-4}\int \frac{x^2 dx}{(1+\frac{\lambda x^2}{\beta^2})^3} \\ &= \beta^{-4}\cdot\frac{\pi}{16}\left(\frac{\lambda}{\beta^2}\right)^{-3/2} \\ &= \frac{\pi}{16\lambda^{3/2}}\beta^{-1}, \end{aligned} \tag{7}$$

where the inequality is due to $e^x-1>x$. This means that when the temperature is higher than the Planck temperature the state equation of the thermal radiantion is essentially different from the well known conclusion, $u \sim \beta^{-4}$. This will influence the emission of the black hole. According to the Stefan-Boltzmann law, the loss mass rate of a Schwarzscild black hole reads

$$\frac{dM}{dt} \sim \beta^{-4} A \sim \frac{1}{M^2}, \tag{8}$$

where M the mass of the hole. At the last stages of emission, $M \to 0$, so the emission rate becomes divergent. However, from Eq. (7), at the last stage, the rate will be changed to

$$\frac{dM}{dt} \sim \beta^{-1} A \sim M \to 0, \tag{9}$$

here is no burst.

We turn to the problem of black hole entropy. Recalling the brick wall model, the number of quantum states less than energy ω is given by[2][11][12]

$$\Gamma(\omega) = \frac{2\omega^3}{3\pi} \int_{r_0+\epsilon}^{L} \frac{dr}{f^2}, \tag{10}$$

which is for a massless scalar field in a spherical and static space-time as follows

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{11}$$

where $f = f(r)$. The horizon is located by $f(r_0) = 0$. ϵ is the cutoff near the horizon. Obviously, the number of states is divergent if $\epsilon = 0$. We carefully check the derivation of Eq. (10) and find that it agrees with Eq. (2), not (3). The former leads to the following formula

$$S = \frac{8\pi^3}{45\beta^3} \int \frac{r^2 dr}{f^2}, \tag{12}$$

which is analogous with the usual state equation of the thermal radiation: $(\beta\sqrt{f})^{-1}$ is the local temperature, $4\pi r^2/\sqrt{f}$ is the proper element of the spatial volume. The divergent entropy means the invalidity of the usual state equation near the black hole horizon. If we take Eq. (3), the situation may be essentially different. Why not have a try? Substituting the wave function $\Phi = \exp(-i\omega t)\psi(r, \theta, \varphi)$ into the equation of massless scalar field

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0, \tag{13}$$

we obtain

$$\frac{\partial^2 \psi}{\partial r^2} + \Big(\frac{f'}{f} + \frac{2}{r}\Big)\frac{\partial \psi}{\partial r} + \frac{1}{f}\left[\frac{\omega^2}{f} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right)\right]\psi = 0. \tag{14}$$

By using the WKB approximation with $\psi \sim \exp[iS(r,\theta,\phi)]$, we have

$$p_r^2 = \frac{1}{f}\left[\frac{\omega^2}{f} - \frac{1}{r^2}p_\theta^2 - \frac{1}{r^2\sin^2\theta}p_\varphi^2\right], \tag{15}$$

where

$$p_r = \frac{\partial S}{\partial r}, p_\theta = \frac{\partial S}{\partial \theta}, p_\varphi = \frac{\partial S}{\partial \varphi}. \tag{16}$$

We also obtain the square module of momentum

$$p^2 = p_i p^i = g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\varphi\varphi}p_\varphi^2 = \frac{\omega^2}{f}. \tag{17}$$

From Eq. (3), the number of quantum states with energy less than ω is given by

$$\begin{aligned}
g(\omega) &= \frac{1}{(2\pi)^3}\int \frac{drd\theta d\varphi dp_r dp_\theta dp_\varphi}{(1+\lambda\omega^2/f)^3} \\
&= \frac{1}{(2\pi)^3}\int \frac{drd\theta d\varphi}{(1+\lambda\omega^2/f)^3}\int \frac{2}{f^{1/2}}\left[\frac{\omega^2}{f} - \frac{1}{r^2}p_\theta^2 - \frac{1}{r^2\sin^2\theta}p_\varphi^2\right]^{1/2} dp_\theta dp_\varphi \\
&= \frac{4\pi\omega^3}{3(2\pi)^3}\int \frac{r^2 dr}{f^2(1+\lambda\omega^2/f)^3}\int \sin\theta d\theta d\varphi \\
&= \frac{2\omega^3}{3\pi}\int \frac{r^2 dr}{f^2(1+\lambda\omega^2/f)^3},
\end{aligned} \tag{18}$$

where the integration goes over those values of p_θ, p_φ for which the argument of the square root is positive, refer to Refs.[2],[11] and [12]. When $\lambda = 0$, Eq. (18) naturally returns to (10). However, in the case $\lambda \neq 0$, Eq. (18) is essentially different from (10): it is convergent at the horizon without any cutoff! By using the usual method, the free energy is given by

$$\begin{aligned}
F(\beta) &= \frac{1}{\beta}\int dg(\omega)\ln(1-e^{-\beta\omega}) \\
&= -\int_0^\infty \frac{g(\omega)d\omega}{e^{\beta\omega}-1} \\
&= -\frac{2}{3\pi}\int_{r_0}\frac{r^2 dr}{f^2}\int_0^\infty \frac{\omega^3 d\omega}{(e^{\beta\omega}-1)(1+\lambda\omega^2/f)^3}.
\end{aligned} \tag{19}$$

The entropy reads

$$\begin{aligned}
S &= \beta^2\frac{\partial F}{\partial \beta} \\
&= \frac{2\beta^2}{3\pi}\int_{r_0}\frac{r^2 dr}{f^2}\int_0^\infty \frac{e^{\beta\omega}\omega^4 d\omega}{(e^{\beta\omega}-1)^2(1+\lambda\omega^2/f)^3} \\
&= \frac{2\beta^{-3}}{3\pi}\int_{r_0}\frac{r^2 dr}{f^2}\int_0^\infty \frac{x^4 dx}{(1-e^{-x})(e^x-1)(1+\frac{\lambda x^2}{\beta^2 f})^3},
\end{aligned} \tag{20}$$

where $x = \beta\omega$. Taking into account the following inequalities

$$\begin{aligned} 1 - e^{-x} &> \frac{x}{1+x}, \\ e^x - 1 &> x, \end{aligned} \tag{21}$$

We obtain

$$\begin{aligned} S &< \frac{2\beta^{-3}}{3\pi} \int_{r_0} \frac{r^2 dr}{f^2} \int_0^\infty \frac{(x^3 + x^2)dx}{(1 + \frac{\lambda x^2}{\beta^2 f})^3} \\ &= \frac{2\beta^{-3}}{3\pi} \int_{r_0} \frac{r^2 dr}{f^2} \left[\frac{1}{4} \left(\frac{\lambda}{\beta^2 f} \right)^{-2} + \frac{\pi}{16} \left(\frac{\lambda}{\beta^2 f} \right)^{-3/2} \right] \\ &= \frac{\beta}{6\pi\lambda^2} \int_{r_0} r^2 dr + \frac{\lambda^{-3/2}}{24} \int_{r_0} \frac{r^2 dr}{f^{1/2}}. \end{aligned} \tag{22}$$

We are only interested in the contribution from the vicinity near the horizon, $[r_0, r_0 + \epsilon]$, which corresponds to a proper distance of order of the minimal length, $2\sqrt{\lambda}$. This is because the entropy close to the upper bound only in this vicinity. Furthermore, it is just the vicinity neglected by brick wall model. We have

$$\begin{aligned} 2\sqrt{\lambda} &= \int_{r_0}^{r_0+\epsilon} \frac{dr}{\sqrt{f}} \\ &\approx \int_{r_0}^{r_0+\epsilon} \frac{dr}{\sqrt{2\kappa(r - r_0)}} \\ &= \sqrt{\frac{2\epsilon}{\kappa}}, \end{aligned} \tag{23}$$

where κ is the surface gravity at the horizon of black hole and it is identified as $\kappa = 2\pi\beta^{-1}$. Thus we naturally derive the entropy proportional to the horizon area

$$\begin{aligned} S &\sim \frac{\beta}{6\pi\lambda^2} r_0^2 \epsilon + \frac{\lambda^{-3/2}}{24} \cdot 2r_0^2 \sqrt{\lambda} \\ &= \frac{3A}{16\lambda\pi}, \end{aligned} \tag{24}$$

where $A = 4\pi r_0^2$ is the surface area of the black hole.

In the earlier 1992, Li and Liu phenomenally proposed that the state equations of the thermal radiation near the horizon should be changed to a series of new formulae rather than Eq. (12), in order to maintain the validity of the generalized second law of thermodynaics[13]. Using the Li-Liu equation, Wang investigates the entropy of a self-gravitational radiation system and obtains the Bekenstein's entropy bound[14]. Here, Parallel to the brick wall model, the scalar field near the horizon of a static black hole is investigated again, we obtain the entropy proportional to the horizon area. There is no divergence without any cutoff near the horizon. This convergency is due to the effect of the generalized uncertainty relation on the quantum states. This provides an evidence for the idea of Li and Liu. The more details between the Li-Liu equation and the generalized uncertainty relation will be investigated in the future.

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